Let $D = (V, A)$ be a digraph. A strong clique of $D$ is a set $C$ of vertices such that for any two distinct vertices $x, y \in C$ both arcs $(x, y), (y, x)$ are in $D$. Let $S, S'$ be two disjoint sets of vertices of $D$: we say that $S$ is completely adjacent to $S'$ (or $S'$ is completely adjacent from $S$) if for any $x \in S, x' \in S'$, the pair $(x, x')$ is an arc of $D$; we say that $S$ is completely non-adjacent to $S'$ (or $S'$ is completely non-adjacent from $S$) if for any $x \in S, x' \in S'$, the pair $(x, x')$ is not an arc of $D$. Let $M$ be a fixed $\{0, 1\}$ matrix of size $m$, with $k$ diagonal 0’s and $\ell$ diagonal 1’s. An $M$-partition of a digraph $D$ is a partition of its vertex set $V(D)$ into parts $V_1, V_2, \ldots, V_{k+\ell}$ such that

- $V_i$ is an independent set of $D$ if $M(i, i) = 0$
- $V_i$ is a strong clique of $D$ if $M(i, i) = 1$
- $V_i$ is completely non-adjacent to $V_j$ if $M(i, j) = 0$
- $V_i$ is completely adjacent to $V_j$ if $M(i, j) = 1$

A full homomorphism of a digraph $D$ to a digraph $H$ is a mapping $f : V(D) \to V(H)$ such that for vertices $x \neq y, (x, y) \in A(D)$ if and only if $(f(x), f(y)) \in A(H)$. If $H$ denote the digraph whose adjacency matrix is $M$, then $D$ admits an $M$-partition if and only if it admits a full homomorphism to $H$.

Undirected graphs are viewed as special cases of digraphs, i.e., each edge $xy$ is viewed as the two arcs $(x, y), (y, x)$. For a symmetric $\{0, 1\}$ matrix $M$, the same definition applies to define an $M$-partition of a graph $G$ [3]. It is shown in [1, 2] that for any symmetric $\{0, 1\}$ matrix $M$ there is a finite set $\mathcal{G}$ of graphs such that $G$ admits an $M$-partition if and only if it does not contain an induced subgraph isomorphic to a member of $\mathcal{G}$. Alternately [3],
we define a minimal obstruction to $M$-partition to be a digraph $D$ which does not admit an $M$-partition, but such that for any vertex $v$ of $D$, the digraph $D - v$ does admit an $M$-partition. Each symmetric $\{0, 1\}$ matrix $M$ has only finitely many minimal graph obstructions [1, 2]. It was known these obstructions have at most $(k + 1)(\ell + 1)$ vertices [2] and this bound is best possible; however, the minimum upper bound has been open for digraphs. We prove that in fact also each minimal digraph obstruction has at most $(k + 1)(\ell + 1)$ vertices (and this is best possible). We interpret our results as certain dualities of full homomorphisms, in the spirit of [1].

A graph is point determining if distinct vertices have distinct neighbourhoods. According to [4], each point determining graph $H$ contains a vertex (in fact, at least two vertices) $v$ such that $H - v$ is also point determining. For the purposes of our proof, we show that every point determining digraph $D$ has a vertex $v$ such that $D - v$ is still point determining.

**Keywords:** matrix partitions, digraph homomorphism, full homomorphism.

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**References**


