We consider directed hypergraphs with hyperarcs of size 3 of the form \( \{u, v\} \rightarrow w \). \( \{u, v\} \) is called the body and \( w \) is called the head of the hyperarc. Let \( H = (V, F) \) be a hypergraph. We say that a vertex \( w \in V \) is reachable from a set \( S \subset V \) if the following process marks \( w \): start by marking vertices in \( S \), and as long as there is a hyperarc \( \{a, b\} \rightarrow c \) such that \( a \) and \( b \) are both marked and \( c \) is unmarked, mark \( c \) as well.

Let \( G = (V, E) \) be a graph. We say that a hypergraph \( H = (V, F) \) represents \( G \) if for every pair \( \{u, v\} \) the set of vertices reachable from \( \{u, v\} \) in \( H \) is the whole vertex set \( V \) if \( uv \in E \) and is \( \{u, v\} \) otherwise. The minimum number of hyperarcs in a hypergraph representing \( G \) is called the hydra number \( h(G) \) of \( G \) and every hypergraph with \( h(G) \) hyperarcs representing \( G \) is called optimal for \( G \). In other words, given a set of hypergraph bodies (the edge set of a given graph), we look for the minimal number of heads assigned to these bodies such that every vertex is reachable from every body.

The problem of finding the hydra number of a graph is related to the minimization problem for Horn formulas in propositional logic.

The authors of [1] proposed the following problem: Let \( G \) consist of \( k \) connected components \( G_1, \ldots, G_k \) for \( k \geq 2 \), such that each \( G_i \) contains at least two vertices and \( s \) of the components are singleheaded. Then does the following hold:

\[
h(G) = \sum_{i=1}^{k} h(G_i) + s.
\]

We prove the upper bound and present some examples of graphs with the hydra number strictly smaller than the upper bound. We also give some sufficient conditions for the equality to hold.

References